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A high order compact MAC finite difference scheme for the Stokes equations: Augmented variable approach

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ABSTRACT

This paper deals with the steady Stokes flow on a rectangular domain. A high order compact MAC finite difference scheme based on the staggered grid is developed for solving Stokes equations with a Dirichlet boundary condition on the velocity. A novel high order boundary treatment is developed via introducing a suitable augmented variable. The accuracy of the proposed method is demonstrated in test problems. Creeping flow solutions for driven cavity problem are obtained numerically and compared with published results.

1. Introduction

In this paper we discuss the higher order MAC (Marker and Cell) finite difference approximations to the Stokes equations in the rectangular domain $\Omega = [a,b] \times [c,d]$ based on the staggered grid as shown Fig. 1. The governed equations are:

$$\begin{aligned} -\Delta\vec{u} + \nabla p &= \vec{f}, \\ \nabla\cdot\vec{u} &= 0, \end{aligned} \tag{1.1}$$

where $\vec{u} = (u,v)^{\mathrm{T}}$ is the velocity, p is the pressure, and $\vec{f} = (f_1,f_2)^{\mathrm{T}}$ is an external force.

A flow that is governed by the Stokes equations is known as a creeping or Stokes flow. Such flow is used for the fluid with very low Reynolds number. That is, the inertia term of the Navier–Stokes equation is very small compared to the viscous forces and it is neglected and results in (1.1). See [6] for a brief introduction to the life at low Reynolds number. We refer [5] for discussions on various methods for solving the viscous incompressible flow at low Reynolds numbers. In [8], the Stokes flows were studied experimentally and computationally for a vortex mixing and multi-cell flows in slender cavities. In [2,3], the finite difference methods based on the standard grid using Poisson equation for the pressure (replacing the divergence free condition) were proposed for the partially or totally periodic boundary condition on the velocity. A similar approach as in this paper is studied to complete the second order pressure boundary condition. A well known finite element scheme for the Stokes equation is the $Q1-L2$ mixed element method. Recently, a $Q1-Q1$ bilinear element was presented in [9]. The inf–sup condition for the pressure space was circumvented. Boundary element method were proposed to solve the Stokes flow in [7,11,12]. By introducing the vorticity $\omega = \nabla\times\vec{u}$, the Stokes flow can also be solved with the vorticity and stream function equations [1,13,14]: $-\Delta\Psi = \omega$, and $\Delta\omega = -\nabla\times\vec{f}$, where Ψ is the stream function. In [15] a Dirichlet

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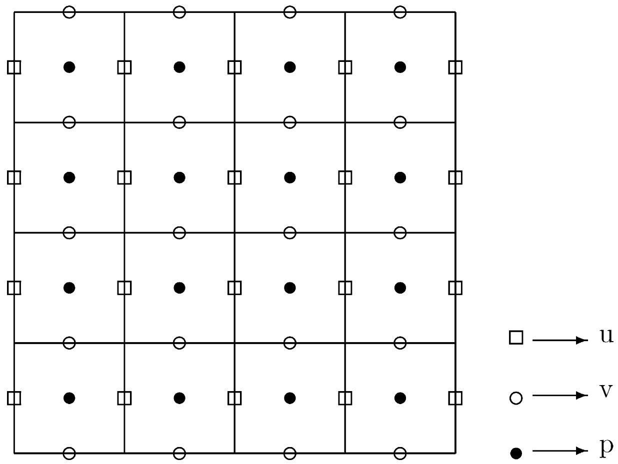


Fig. 1. Staggered grid.

boundary treatment for the second order MAC scheme is developed. Our second order boundary treatment results in the method in [15] based on a different approach. The approach we used is extended to the fourth order naturally.

In this paper, we develop a compact fourth order MAC finite difference scheme for (1.1) with periodic boundary conditions as well as the Dirichlet boundary conditions. Extension of our approach to the 3-dimensional case is straightforward. In general, the compact high order schemes are developed using the continuation of solutions by the Taylor series expansion and the determination of the Taylor coefficients via differentiating equations and boundary conditions, see [18]. It has a local truncation error of order h^4 and is an approximation of compact type as in 2D case it involves eight nearest neighbors of the point (x, y) about which the differences are taken (hence, the scheme is called compact). This definition of compactness differs from the definition given in [19]. Such technique has been applied to a variety of elliptic equations, see [25,26,28,29]. Many compact schemes have been well developed for Stokes problems based on the vorticity and stream function equations, see [20–23,27,30,31].

Our compact scheme is developed for the velocity–pressure formulations of the Stokes equations. The compact scheme is advantageous since it yields highly accurate numerical solutions and no extra boundary conditions are needed to complete the system, especially for the divergence free condition. The compact scheme results in matrices with not only much smaller band-width but also with M property. An advantage of the staggered grid shown in Fig. 1 is that there is no need for pressure boundary conditions dealing with ∇p since the pressure nodes are the center of the MAC cell. But we must treat the Dirichlet boundary condition since some of the velocity nodes are not on the boundary. We introduce an augmented variable at the boundary and develop a higher order boundary treatment. Such technique is to set $g = \frac{\partial}{\partial \nu}(\nu \cdot \vec{u})$ as part of the unknowns, which called the augmented variable, along the boundary, which should satisfy the momentum equation projected to the same boundary, see [2,3]. In [17], an alternative compact fourth order method is developed based on a staggered grid with pressure nodes at the grid-points.

An outline of the paper is as follows. In Section 2 we develop the compact fourth order scheme for the case of periodic boundary conditions. In Section 3 we discuss the case of Dirichlet boundary conditions. We use boundary flux as augmented variables to complete the system. In Section 4 we present numerical validation of the proposed schemes. Driven cavity flows in rectangle domains are used to verify the feasibility of the method.

2. Periodic boundary condition

In this section we consider (1.1) with periodic boundary conditions in the square $\Omega = [0, 1] \times [0, 1]$.

2.1. Second order method

Let $\vec{u} = (u, v)$ and $h = \frac{1}{N}$. The standard second order method is defined by

$$\begin{aligned}
&-\frac{u^{i+1,j+\frac{1}{2}} - 2u^{i,j+\frac{1}{2}} + u^{i-1,j+\frac{1}{2}}}{h^2} - \frac{u^{i,j+\frac{3}{2}} - 2u^{i,j+\frac{1}{2}} + u^{i,j-\frac{1}{2}}}{h^2} + \frac{p^{i+\frac{1}{2},j+\frac{1}{2}} - p^{i-\frac{1}{2},j+\frac{1}{2}}}{h} = f_1^{i,j+\frac{1}{2}},\\
&-\frac{v^{i+\frac{3}{2},j} - 2v^{i+\frac{1}{2},j} + v^{i-\frac{1}{2},j}}{h^2} - \frac{v^{i+\frac{1}{2},j+1} - 2v^{i+\frac{1}{2},j} + v^{i+\frac{1}{2},j-1}}{h^2} + \frac{p^{i+\frac{1}{2},j+\frac{1}{2}} - p^{i+\frac{1}{2},j-\frac{1}{2}}}{h} = f_2^{i+\frac{1}{2},j},\\
&\frac{u^{i+1,j+\frac{1}{2}} - u^{i,j+\frac{1}{2}}}{h} + \frac{v^{i+\frac{1}{2},j+1} - v^{i+\frac{1}{2},j}}{h} = 0,
\end{aligned} \tag{2.1}$$

with periodic conditions

$$u^{i+N,j+\frac{1}{2}+N}=u^{i,j+\frac{1}{2}},\quad v^{i+\frac{1}{2}+N,j+N}=v^{i+\frac{1}{2},j},\quad p^{i+\frac{1}{2}+N,j+\frac{1}{2}+N}=p^{i+\frac{1}{2},j+\frac{1}{2}},$$

for all $0\leqslant i,j\leqslant N-1$. Here

$$u^{i,j+\frac{1}{2}}\sim u\left(ih,\left(j+\frac{1}{2}\right)h\right),$$
$$v^{i+\frac{1}{2},j}\sim v\left(\left(i+\frac{1}{2}\right)h,jh\right),$$
$$p^{i+\frac{1}{2},j+\frac{1}{2}}\sim p\left(\left(i+\frac{1}{2}\right)h,\left(j+\frac{1}{2}\right)h\right).$$

System (2.1) is written in the form

$$\begin{aligned} &Hu-B_1^{\mathrm{T}}p=f_1,\\ &Hv-B_2^{\mathrm{T}}p=f_2,\\ &B_1u+B_2v=0. \end{aligned}\tag{2.2}$$

The symmetric matrix $H=H^0\otimes I+I\otimes H^0$, where $\otimes$ is the Kronecker product and the symmetric matrix $H^0\in R^{N\times N}$ has non-zero entries:

$$H^0_{i,i}=2N^2,\quad H^0_{i,i-1}=H^0_{i-1,i}=-N^2\quad\text{and}\quad H_{N,1}=H_{1,N}=-N^2.$$

And $B_1=D^0\otimes I$, $B_2=I\otimes D^0$ with the difference matrix $D^0\in R^{N\times N}$:

$$D^0_{i,i+1}=N,\quad D^0_{i,i}=-N,\quad D^0_{N,1}=N.$$

Also, note that $B_1H=HB_1$ and $B_2H=HB_2$. Thus, (2.2) is equivalently written as

$$\begin{aligned} &Hu-B_1^{\mathrm{T}}p=f_1,\\ &Hv-B_2^{\mathrm{T}}p=f_2,\\ &-Hp=B_1f_1+B_2f_2, \end{aligned}\tag{2.3}$$

where we used $H=B_1B_1^{\mathrm{T}}+B_2B_2^{\mathrm{T}}$. That is, the pressure is determined by solving a Poisson equation, separately.

2.2. Compact fourth order method

In this section we develop the compact fourth order difference scheme for the periodic boundary condition. Note that

$$\frac{\phi(x+\frac{h}{2},y)-\phi(x-\frac{h}{2},y)}{h}=\phi_x(x,y)+\frac{h^2}{24}\phi_{xxx}(x,y)+O(h^4).\tag{2.4}$$

We apply (2.4) for $\phi=p$ at $(ih,(j+\frac{1}{2})h)$ and note that since $p_{xx}+p_{yy}=\nabla\cdot\vec{f}\equiv F$,

$$p_{xxx}=F_x-p_{xyy},\quad p_{yyy}=F_y-p_{xxy}.$$

Here we approximate p_{xyy} by

$$p_{xyy}(ih,\left(j+\frac{1}{2}\right)h)\sim\left(\frac{p^{i+\frac{1}{2},j+\frac{3}{2}}-2p^{i+\frac{1}{2},j+\frac{1}{2}}+p^{i+\frac{1}{2},j-\frac{1}{2}}}{h^2}-\frac{p^{i-\frac{1}{2},j+\frac{3}{2}}-2p^{i-\frac{1}{2},j+\frac{1}{2}}+p^{i-\frac{1}{2},j-\frac{1}{2}}}{h^2}\right)\frac{1}{h},\tag{2.5}$$

in the second order. Thus we obtain the fourth order difference of $p_x(ih,(j+\frac{1}{2})h)$:

$$p_x\left(ih,\left(j+\frac{1}{2}\right)h\right)\sim\frac{p^{i+\frac{1}{2},j+\frac{1}{2}}-p^{i-\frac{1}{2},j+\frac{1}{2}}}{h}+\frac{1}{24h}\left(\left(p^{i+\frac{1}{2},j+\frac{3}{2}}-2p^{i+\frac{1}{2},j+\frac{1}{2}}+p^{i+\frac{1}{2},j-\frac{1}{2}}\right)-\left(p^{i-\frac{1}{2},j+\frac{3}{2}}-2p^{i-\frac{1}{2},j+\frac{1}{2}}+p^{i-\frac{1}{2},j-\frac{1}{2}}\right)\right)-\frac{h^2}{24}F_x^{i,j+\frac{1}{2}}.$$

Similarly, we have

$$p_y\left(\left(i+\frac{1}{2}\right)h,jh\right)\sim\frac{p^{i+\frac{1}{2},j+\frac{1}{2}}-p^{i+\frac{1}{2},j-\frac{1}{2}}}{h}+\frac{1}{24h}\left(\left(p^{i+\frac{3}{2},j+\frac{1}{2}}-2p^{i+\frac{1}{2},j+\frac{1}{2}}+p^{i-\frac{1}{2},j+\frac{1}{2}}\right)-\left(p^{i+\frac{3}{2},j-\frac{1}{2}}-2p^{i+\frac{1}{2},j-\frac{1}{2}}+p^{i-\frac{1}{2},j-\frac{1}{2}}\right)\right)-\frac{h^2}{24}F_y^{i+\frac{1}{2},j}.$$

For the Laplacian terms note that

$$\frac{u^{i+1,j+\frac{1}{2}}-2u^{i,j+\frac{1}{2}}+u^{i-1,j+\frac{1}{2}}}{h^2}+\frac{u^{i,j+\frac{3}{2}}-2u^{i,j+\frac{1}{2}}+u^{i,j-\frac{1}{2}}}{h^2}=u_{xx}+u_{yy}+\frac{h^2}{12}(u_{xxxx}+u_{yyyy})+O(h^4),\tag{2.6}$$

and

$$u_{xxxx}+u_{yyyy}=(p_{xx}+p_{yy})_x-((f_1)_{xx}+(f_1)_{yy})-2u_{xxyy}.\tag{2.7}$$

Here u_{xxyy} can be approximated in the second order based on the standard 9-point stencils at $(ih, j+\frac{1}{2}h)$ by

$$h^2 u_{xxyy} \sim \frac{u^{i+1,j+\frac{3}{2}} - 2u^{i,j+\frac{3}{2}} + u^{i-1,j+\frac{3}{2}}}{h^2} - 2\frac{u^{i+1,j+\frac{1}{2}} - 2u^{i,j+\frac{1}{2}} + u^{i-1,j+\frac{1}{2}}}{h^2} + \frac{u^{i+1,j-\frac{1}{2}} - 2u^{i,j-\frac{1}{2}} + u^{i-1,j-\frac{1}{2}}}{h^2}.$$

Thus, we obtain the fourth order difference scheme for u-equation:

$$\begin{aligned}&\left(\frac{10}{3}u^{i,j+\frac{1}{2}} - \frac{2}{3}\left(u^{i,j-\frac{1}{2}} + u^{i-1,j+\frac{1}{2}} + u^{i+1,j+\frac{1}{2}} + u^{i,j+\frac{3}{2}}\right) - \frac{1}{6}\left(u^{i-1,j-\frac{1}{2}} + u^{i-1,j+\frac{3}{2}} + u^{i+1,j-\frac{1}{2}} + u^{i+1,j+\frac{3}{2}}\right)\right)\frac{1}{h^2} + \frac{p^{i+\frac{1}{2},j+\frac{1}{2}} - p^{i-\frac{1}{2},j+\frac{1}{2}}}{h}\\ &\quad + \frac{1}{24h}\left(\left(p^{i+\frac{1}{2},j+\frac{3}{2}} - 2p^{i+\frac{1}{2},j+\frac{1}{2}} + p^{i+\frac{1}{2},j-\frac{1}{2}}\right) - \left(p^{i-\frac{1}{2},j+\frac{3}{2}} - 2p^{i-\frac{1}{2},j+\frac{1}{2}} + p^{i-\frac{1}{2},j-\frac{1}{2}}\right)\right)\\ &\quad + \frac{h^2}{24}F_x^{i,j+\frac{1}{2}} = \frac{2}{3}f_1^{i,j+\frac{1}{2}} + \frac{1}{12}\left(f_1^{i-1,j+\frac{1}{2}} + f_1^{i+1,j+\frac{1}{2}} + f_1^{i,j+\frac{3}{2}} + f_1^{i,j-\frac{1}{2}}\right).\end{aligned} \tag{2.8}$$

Similarly, for v-equation:

$$\begin{aligned}&\left(\frac{10}{3}v^{i+\frac{1}{2},j} - \frac{2}{3}\left(v^{i-\frac{1}{2},j} + v^{i+\frac{1}{2},j-1} + v^{i+\frac{1}{2},j+1} + v^{i+\frac{3}{2},j}\right) - \frac{1}{6}\left(v^{i-\frac{1}{2},j-1} + v^{i-\frac{1}{2},j+1} + v^{i+\frac{3}{2},j-1} + v^{i+\frac{3}{2},j+1}\right)\right)\frac{1}{h^2}\\ &\quad + \frac{p^{i+\frac{1}{2},j+\frac{1}{2}} - p^{i+\frac{1}{2},j-\frac{1}{2}}}{h} + \frac{1}{24h}\left(\left(p^{i+\frac{3}{2},j+\frac{1}{2}} - 2p^{i+\frac{1}{2},j+\frac{1}{2}} + p^{i-\frac{1}{2},j+\frac{1}{2}}\right) - \left(p^{i+\frac{3}{2},j-\frac{1}{2}} - 2p^{i+\frac{1}{2},j-\frac{1}{2}} + p^{i-\frac{1}{2},j-\frac{1}{2}}\right)\right)\\ &\quad + \frac{h^2}{24}F_y^{i+\frac{1}{2},j} = \frac{2}{3}f_2^{i+\frac{1}{2},j} + \frac{1}{12}\left(f_2^{i-\frac{1}{2},j} + f_2^{i+\frac{1}{2},j-1} + f_2^{i+\frac{1}{2},j+1} + f_2^{i+\frac{3}{2},j}\right).\end{aligned} \tag{2.9}$$

For the divergence free condition we note that

$$u_{xxx} = -v_{xxy}, \quad v_{yyy} = -u_{xyy}.$$

Using the same procedure as above, we have the fourth order difference scheme for the divergence free condition:

$$\begin{aligned}&\frac{u^{i+1,j+\frac{1}{2}} - u^{i,j+\frac{1}{2}}}{h} + \frac{1}{24h}\left(\left(u^{i+1,j+\frac{3}{2}} - 2u^{i+1,j+\frac{1}{2}} + u^{i+1,j-\frac{1}{2}}\right) - \left(u^{i,j+\frac{3}{2}} - 2u^{i,j+\frac{1}{2}} + u^{i,j-\frac{1}{2}}\right)\right) + \frac{v^{i+\frac{1}{2},j+1} - v^{i+\frac{1}{2},j}}{h}\\ &\quad + \frac{1}{24h}\left(\left(v^{i+\frac{3}{2},j+1} - 2v^{i+\frac{1}{2},j+1} + v^{i-\frac{1}{2},j+1}\right) - \left(v^{i+\frac{3}{2},j} - 2v^{i+\frac{1}{2},j} + v^{i-\frac{1}{2},j}\right)\right) = 0.\end{aligned} \tag{2.10}$$

The local matrix representation of approximations to $-\Delta$, $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, and the mass matrix are given by

$$\frac{1}{h^2}\begin{pmatrix} -\frac{1}{6} & -\frac{2}{3} & -\frac{1}{6} \\ -\frac{2}{3} & \frac{10}{3} & -\frac{2}{3} \\ -\frac{1}{6} & -\frac{2}{3} & -\frac{1}{6} \end{pmatrix}, \quad \frac{1}{h}\begin{pmatrix} -\frac{1}{24} & \frac{1}{24} \\ -\frac{11}{12} & \frac{11}{12} \\ -\frac{1}{24} & \frac{1}{24} \end{pmatrix}, \quad \frac{1}{h}\begin{pmatrix} \frac{1}{24} & \frac{11}{12} & \frac{1}{24} \\ -\frac{1}{24} & -\frac{11}{12} & -\frac{1}{24} \end{pmatrix}, \quad \begin{pmatrix} 0 & \frac{1}{12} & 0 \\ \frac{1}{12} & \frac{2}{3} & \frac{1}{12} \\ 0 & \frac{1}{12} & 0 \end{pmatrix}. \tag{2.11}$$

System (2.8)–(2.10) is written in the form

$$\begin{aligned}&Hu - B_1^{\mathrm{T}}p = Qf_1\\ &Hv - B_2^{\mathrm{T}}p = Qf_2\\ &B_1u + B_2v = 0,\end{aligned}$$

where Q, the mass matrix, and the corresponding matrices (H, B_1, B_2) are as described above. Again we have $HB_1 = B_1H$ and $HB_2 = B_2H$ and we have

$$\begin{aligned}&Hu - B_1^{\mathrm{T}}p = Qf_1,\\ &Hv - B_2^{\mathrm{T}}p = Qf_2,\\ &-\widehat{H}p = B_1Qf_1 + B_2Qf_2,\end{aligned} \tag{2.12}$$

where $\widehat{H} = B_1B_1^{\mathrm{T}} + B_2B_2^{\mathrm{T}}$. Hence we have

$$(Hu_1, u_2) + (Hu_2, u_2) = (Qf_1, u_1) + (Qf_2, u_2), \quad -(\widehat{H}p, p) = (B_1^{\mathrm{T}}p, Qf_1) + (B_2^{\mathrm{T}}p, Qf_2), \tag{2.13}$$

which shows the stability of system (2.8)–(2.10) and provides an error estimate using the standard analysis.

3. Dirichlet boundary condition

In this section we consider (1.1) with no-slip Dirichlet boundary conditions on velocities in the square domain $\Omega = [0,1] \times [0,1]$:

$$\begin{aligned}&-\Delta\vec{u} + \nabla p = \vec{f} \quad \text{in} \quad \Omega\\ &\nabla \cdot \vec{u} = 0 \quad \text{in} \quad \Omega,\\ &\vec{u} = 0, \quad \text{on} \quad \partial\Omega.\end{aligned} \tag{3.1}$$

This system is proven to be well-posed in [24].

3.1. Second order method

We use augmented variables $\frac{\partial u}{\partial v} = g_1$ at the sides $y = 0$ and $y = 1$ with $0 \leqslant x \leqslant 1$. Then at $y = \frac{h}{2}$, we have

$$\frac{\partial^2 u}{\partial y^2}\left(ih, \frac{h}{2}\right) = \frac{\frac{\partial u}{\partial y}(ih,h) - \frac{\partial u}{\partial y}(ih,0)}{h} + O(h) = \frac{\frac{u^{i,\frac{3}{2}} - u^{i,\frac{1}{2}}}{h} - g_1^{i,0}}{h} + O(h).$$

Similarly, at $y = 1 - \frac{h}{2}$, we have

$$\frac{\partial^2 u}{\partial y^2}\left(ih, 1 - \frac{h}{2}\right) = \frac{\frac{u^{i,\frac{3}{2}} - u^{i,\frac{1}{2}}}{h} - g_1^{i,N}}{h} + O(h).$$

So for $1 \leqslant i \leqslant N-1$, we can have the first order difference scheme for the Laplacian term of u at $y = \frac{h}{2}$ and $y = 1 - \frac{h}{2}$:

$$-\frac{u^{i,\frac{3}{2}} - u^{i,\frac{1}{2}}}{h^2} - \frac{g_1^{i,0}}{h} - \frac{u^{i+1,\frac{1}{2}} - 2u^{i,\frac{1}{2}} + u^{i-1,\frac{1}{2}}}{h^2} + \frac{p^{i+\frac{1}{2},\frac{1}{2}} - p^{i-\frac{1}{2},\frac{1}{2}}}{h} = f_1^{i,\frac{1}{2}}, \tag{3.2}$$

$$\frac{u^{i,N-\frac{1}{2}} - u^{i,N-\frac{3}{2}}}{h^2} - \frac{g_1^{i,N}}{h} - \frac{u^{i+1,N-\frac{1}{2}} - 2u^{i,N-\frac{1}{2}} + u^{i-1,N-\frac{1}{2}}}{h^2} + \frac{p^{i+\frac{1}{2},N-\frac{1}{2}} - p^{i-\frac{1}{2},N-\frac{1}{2}}}{h} = f_1^{i,N-\frac{1}{2}}. \tag{3.3}$$

Otherwise, for $1 \leqslant j \leqslant N-2$, we have the second order difference scheme for the Laplacian term of u:

$$-\frac{u^{i+1,j+\frac{1}{2}} - 2u^{i,j+\frac{1}{2}} + u^{i-1,j+\frac{1}{2}}}{h^2} - \frac{u^{i,j+\frac{3}{2}} - 2u^{i,j+\frac{1}{2}} + u^{i,j-\frac{1}{2}}}{h^2} + \frac{p^{i+\frac{1}{2},j+\frac{1}{2}} - p^{i-\frac{1}{2},j+\frac{1}{2}}}{h} = f_1^{i,j+\frac{1}{2}}. \tag{3.4}$$

Here, $u^{0,j+\frac{1}{2}} = u^{N,j+\frac{1}{2}} = 0, \quad 0 \leqslant j \leqslant N-1.$

Suppose we treat the boundary conditions

$$u_y(x,0) = u_y(x,1) = 0, \quad 0 < x < 1,$$

then we set $g_1^{i,0} = g_1^{i,N} = 0$ and the equation is completed for u. For the Dirichlet boundary condition we use the boundary condition $u(x,0) = u(x,1) = 0$ to complete the system. The first order Taylor series approximation of u at $y = 0$ and $y = 1$ results in

$$u^{i,\frac{1}{2}} + g_1^{i,0}\frac{h}{2} = 0, \quad u^{i,N-\frac{1}{2}} + g_1^{i,N}\frac{h}{2} = 0.$$

Similarly, for v-equation we use augmented variable $\frac{\partial v}{\partial v} = g_2$ at the sides $x = 0$ and $x = 1$ with $0 \leqslant y \leqslant 1$.

For the Laplacian term of u, we have

$$A_1 = H^2 \otimes I + I \otimes H^1,$$

where the tridiagonal matrices H^1 and H^2 are given by

$$H^1 = N^2 \begin{pmatrix} 3 & -1 & & & \\ -1 & 2 & -1 & & \\ & \cdot & \cdot & \cdot & \\ & & -1 & 2 & -1 \\ & & & -1 & 3 \end{pmatrix} \in R^{N,N},$$

$$H^2 = N^2 \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \cdot & \cdot & \cdot & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \in R^{N-1,N-1}.$$

Similarly, for the Laplacian term of v we have

$$A_2 = H^1 \otimes I + I \otimes H^2.$$

For the divergence free condition $\nabla \cdot \vec{u} = 0$, we have

$$B_1 u + B_2 v = 0,$$

with $B_1 = D \otimes I, \quad B_2 = I \otimes D$, where the bi-diagonal matrix D is given by

$$D = N \begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ & \cdot & \cdot & \\ & & -1 & 1 \\ & & & -1 \end{pmatrix} \in R^{N-1,N-1}.$$

For $-\nabla p$, we have

$$[-C_1 p \quad -C_2 p],$$

with $C_1 = D^{\mathrm{T}} \otimes I, \quad C_2 = I \otimes D^{\mathrm{T}}$.

So, the second order scheme for the Stokes system with Dirichlet boundary on the velocity can be written as

$$\begin{pmatrix} A_1 & 0 & C_1 \\ 0 & A_2 & C_2 \\ B_1 & B_2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ -p \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ 0 \end{pmatrix}. \tag{3.5}$$

We will reduce (3.5) into the form (2.3) by noting that

$$A_3 = BC = H^3 \otimes I + I \otimes H^3,$$

where $B = (B_1, B_2), C = (C_1, C_2)^{\mathrm{T}}$, and the tri-diagonal matrix H^3 is given by

$$H^3 = N^2 \begin{pmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \cdot & \cdot & \cdot & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{pmatrix} \in R^{N-1,N-1},$$

and corresponds to the central difference approximation to $-\Delta p$ with the Neumann boundary conditions on the square. It can be shown that

$$B_1 A_1 u + B_2 A_2 v = \Delta_1 u + \Delta_2 v,$$

with

$$\Delta_1 = 2N^3 I \otimes O, \quad \Delta_2 = 2N^3 O \otimes I,$$

where $O \in R^{N,N-1}$ has two nonzero elements $O_{11} = -1$ and $O_{N,N-1} = 1$. Thus (3.5) is equivalently to

$$\begin{pmatrix} A_1 & 0 & C_1 \\ 0 & A_2 & C_2 \\ \Delta_1 & \Delta_2 & A_3 \end{pmatrix} \begin{pmatrix} u \\ v \\ -p \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ B_1 f_1 + B_2 f_2 \end{pmatrix}. \tag{3.6}$$

The third equation of (3.6) is equivalent to the pressure boundary condition for the pressure Poisson equation $-\Delta p = \nabla \cdot \vec{f}$. That is, by the Taylor approximation, we have

$$u(h,y) \sim u(0,y) + u_x(0,y)h + \frac{h^2}{2} u_{xx}(0,y),$$

where

$$u(0,y) = u_x(0,y) = 0, \quad u_{xx}(0,y) = p_x(0,y).$$

Thus,

$$p_x(0) = \frac{2}{h^2} u(h,y).$$

3.2. Efficient algorithm

We let

$$A = \begin{pmatrix} A_1 & 0 & C_1 \\ 0 & A_2 & C_2 \\ B_1 & B_2 & 0 \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} \widehat{A}_1 & 0 & C_1 \\ 0 & \widehat{A}_2 & C_2 \\ B_1 & B_2 & 0, \end{pmatrix}$$

where

$$\widehat{A}_1 = H^3 \otimes I + I \otimes H^2, \quad \widehat{A}_2 = H^2 \otimes I + I \otimes H^3.$$

Then, $E(u, v, -p)^{\mathrm{T}} = (f_1, f_2, 0)^{\mathrm{T}}$ is equivalently written as

$$\begin{pmatrix} \widehat{A}_1 & 0 & C_1 \\ 0 & \widehat{A}_2 & C_2 \\ 0 & 0 & A_3 \end{pmatrix} \begin{pmatrix} u \\ v \\ -p \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ B_1 f_1 + B_2 f_2, \end{pmatrix}$$

and it can be solved by the FFT-based method since the eigen-vectors of $\widehat{A}_1$, $\widehat{A}_2$ and A_3 are given by

$$\left\{\cos\left(k\pi\frac{i-\frac{1}{2}}{N}\right)\sin\left(\ell\pi\frac{j}{N}\right)\right\}, \quad \left\{\sin\left(k\pi\frac{i}{N}\right)\cos\left(\ell\pi\frac{j-\frac{1}{2}}{N}\right)\right\}, \quad \left\{\cos\left(k\pi\frac{i-\frac{1}{2}}{N}\right)\cos\left(\ell\pi\frac{j-\frac{1}{2}}{N}\right)\right\},$$

respectively.

Consider the post-conditioned system $AE^{-1}\bar{x} = b$ with $E^{-1}\bar{x} = (u, v, -p)^{\mathrm{T}}$ and $b = (f_1, f_2, 0)^{\mathrm{T}}$. Note that if $\hat{x} = \bar{x} - b$, then

$$AE^{-1}\hat{x} = b - AE^{-1}b = -(A - E)E^{-1}b = \hat{b} \in X = \text{range} \quad (A - E),$$

where we used $AE^{-1} = (A - E)E^{-1} + I$. Thus, $\hat{x} \in X$ satisfies the reduced order system [16] in X:

$$(A - E)E^{-1}\hat{x} + \hat{x} = \hat{b}, \tag{3.7}$$

with

$$A - E = \begin{pmatrix} A_1 - \widehat{A}_1 & 0 & 0 \\ 0 & A_2 - \widehat{A}_2 & 0 \\ 0 & 0 & 0, \end{pmatrix}$$

and $\dim(X) = 4(N - 1)$. That is, (3.7) provides an efficient method to solve (3.5) [16].

3.3. Convergence analysis

Let

$$U^{i,j+\frac{1}{2}} = \frac{1}{h}\int_{jh}^{(j+1)h} u(ih, y)\,\mathrm{d}y,$$
$$V^{i+\frac{1}{2},j} = \frac{1}{h}\int_{ih}^{(i+1)h} v(x, jh)\,\mathrm{d}x.$$

Integrating

$$\frac{1}{h^2}\int_{ih}^{(i+1)h}\int_{jh}^{(j+1)h}(u_x + v_y)\,\mathrm{d}x\mathrm{d}y = 0,$$

we have

$$\frac{U^{i+1,j+\frac{1}{2}} - U^{i,j+\frac{1}{2}}}{h} + \frac{V^{i+\frac{1}{2},j+1} - V^{i+\frac{1}{2},j}}{h} = 0.$$

Let

$$U(x, y) = \frac{1}{h}\int_{y-\frac{h}{2}}^{y+\frac{h}{2}} u(x, s)\,\mathrm{d}s.$$

Then U satisfies

$$-(U_{xx} + U_{yy}) + \left(\int_{y-\frac{1}{2}}^{y+\frac{h}{2}} p(x, s)\,\mathrm{d}s\right)_x = \int_{y-\frac{1}{2}}^{y+\frac{h}{2}} f_1(x, s)\,\mathrm{d}s,$$

with

$$U_{yy} = \frac{u_y(\cdot, h) - u_y(\cdot, 0)}{h} \quad \text{at} \quad y = \frac{h}{2}.$$

Thus, we have the truncation error at $(i, j + \frac{1}{2}), j \neq 0, N - 1$,

$$E^{i,j+\frac{1}{2}} \sim h^2 \epsilon^{i,j+\frac{1}{2}},$$

with

$$\epsilon^{i,j+\frac{1}{2}} = \frac{1}{12}(U_{xxxx} + U_{yyyy}) + \frac{1}{24}h(p_{xx} + p_{yy})_x \quad \text{at} \quad \left(ih, \left(j + \frac{1}{2}\right)h\right),$$

for

$$(\delta u)^{i,j+\frac{1}{2}} = u^{i,j+\frac{1}{2}} - U^{i,j+\frac{1}{2}}, \quad \delta p^{i+\frac{1}{2},j+\frac{1}{2}} = p^{i+\frac{1}{2},j+\frac{1}{2}} - p\left(\left(i + \frac{1}{2}\right)h, \left(j + \frac{1}{2}\right)h\right).$$

At $(ih, \frac{h}{2})$, note that

$$U^{i,\frac{1}{2}} = \frac{1}{h}\int_0^h u(ih, y)\,dy = \frac{h}{2}u_y(ih, 0) + \frac{h^2}{6}u_{yy}(ih, 0) + O(h^3).$$

Thus we have the truncation error at $(i, \frac{1}{2})$

$$E^{i,\frac{1}{2}} \sim h^2\epsilon^{i,\frac{1}{2}} + h^2\frac{1}{3}u_{yy}(ih, 0),$$

with

$$\epsilon^{i,j+\frac{1}{2}} = \frac{1}{12}(U_{xxxx} + u_{yyyy}) + \frac{1}{24}h(p_{xx} + p_{yy})_x \quad \text{at} \quad \left(ih, \left(j + \frac{1}{2}\right)h\right).$$

Similarly, we have the truncation error analysis at $(i, N - \frac{1}{2})$ and for

$$(\delta v)^{i+\frac{1}{2},j} = v^{i+\frac{1}{2},j} - V^{i+\frac{1}{2},j}.$$

In summary the error $(\delta u, \delta v, \delta p)$ satisfies (3.5) with the corresponding truncation errors. Multiplying $(\delta u, \delta v, \delta p)$, we obtain

$$\begin{aligned}\Phi(\delta u, \delta v) &= \sum\sum h^2(|D_x^{i,j+\frac{1}{2}}u|^2 + |D_y^{i,j+\frac{1}{2}}u|^2 + |D_x^{i+\frac{1}{2},j}v|^2 + |D_y^{i+\frac{1}{2},j}v|^2) + \sum(|u^{i,\frac{1}{2}}|^2 + |u^{i,N-\frac{1}{2}}|^2) + \sum(|v^{\frac{1}{2},j}|^2 + |v^{N-\frac{1}{2},j}|^2) \\ &= \sum h^4(\epsilon^{i,j+\frac{1}{2}}u^{i,j+\frac{1}{2}} + \epsilon^{i+\frac{1}{2},j}v^{i+\frac{1}{2},j}) + \sum h^2(u_{yy}(ih, 0)u^{i,\frac{1}{2}} + u_{yy}(ih, 1)u^{i,N-\frac{1}{2}}) + \sum h^2(v_{xx}(0, jh)v^{\frac{1}{2},j} + v_{xx}(1, jh)v^{N-\frac{1}{2},j}),\end{aligned}$$

where

$$D_x^{i,j+\frac{1}{2}}u = \frac{u^{i,j+\frac{1}{2}} - u^{i-1,j+\frac{1}{2}}}{h} \quad D_y^{i,j+\frac{1}{2}}u = \frac{u^{i,j+\frac{1}{2}} - u^{i,j-\frac{1}{2}}}{h}$$
$$D_x^{i+\frac{1}{2},j}v = \frac{u^{i+\frac{1}{2},j} - u^{i-\frac{1}{2},j}}{h} \quad D_y^{i+\frac{1}{2},j}v = \frac{v^{i+\frac{1}{2},j} - v^{i+\frac{1}{2},j-1}}{h}.$$

Hence

$$\sqrt{\Phi(\delta u, \delta p)} \leqslant Ch^2\sqrt{\sum\sum h^2|\epsilon^{i,j+\frac{1}{2}}|^2 + |\epsilon^{i+\frac{1}{2},j}|^2} + \sqrt{\sum(|u_{yy}(ih, 0)|^2 + |u_{yy}(ih, 1)|^2) + \sum(|v_{xx}(0, jh)|^2 + |v_{xx}(1, jh)|^2)}.$$

3.4. Fourth order method for Dirichlet boundary condition

We first have an error estimate at $y = \frac{h}{2}$ and $x = ih, 1 \leqslant i \leqslant N - 1$ for the second order difference (3.2)

$$\frac{\frac{u^{i,\frac{3}{2}} - u^{i,\frac{1}{2}}}{h} + g_1^{i,0}}{h} + \frac{u^{i+1,\frac{1}{2}} - 2u^{i,\frac{1}{2}} + u^{i-1,\frac{1}{2}}}{h^2} = u_{xx}(x, y) + u_{yy}(x, y) + \frac{h}{24}u_{yyy}(x, 0) + \frac{h^2}{12}(u_{xxxx}(x, y) + u_{yyyy}(x, y)) + O(h^3). \tag{3.8}$$

Here, we used

$$\frac{\frac{u^{i,\frac{3}{2}} - u^{i,\frac{1}{2}}}{h} + g_1^{i,0}}{h} = u_{yy}(x, y) + \frac{h}{24}u_{yyy}(x, y) + \frac{3h^2}{48}u_{yyyy}(x, y) + O(h^3),$$

and

$$u_{yyy}(x, 0) = u_{yyy}(x, y) - \frac{h}{2}u_{yyyy}(x, y) + O(h^2).$$

From (2.7), we have

$$u_{xxxx} + u_{yyyy} = -2u_{xxyy} + F_x - \Delta f_1,$$

with the first order approximation

$$h^2u_{xxyy}\left(x, \frac{h}{2}\right) \sim \frac{u^{i+1,\frac{3}{2}} - 2u^{i,\frac{3}{2}} + u^{i-1,\frac{3}{2}}}{h^2} - 2\frac{u^{i+1,\frac{1}{2}} - 2u^{i,\frac{1}{2}} + u^{i-1,\frac{1}{2}}}{h^2} + h(g_1)_{xx}(x, 0).$$

Note that

$$\frac{p^{i+\frac{1}{2},\frac{1}{2}} - p^{i-\frac{1}{2},\frac{1}{2}}}{h} = p_x(x,\frac{h}{2}) + \frac{h^2}{24} p_{xxx}\left(x,\frac{h}{2}\right) + O(h^4),$$

with

$$-p_{xxx}\left(x,\frac{h}{2}\right) = p_{xyy}\left(x,\frac{h}{2}\right) - F_x\left(x,\frac{h}{2}\right) = \frac{\frac{p^{i+\frac{1}{2},\frac{3}{2}} - p^{i+\frac{1}{2},\frac{1}{2}} - \left(p^{i+\frac{1}{2},\frac{3}{2}} - p^{i+\frac{1}{2},\frac{3}{2}}\right)}{h^2} - p_{xy}(x,0)}{h} - F_x\left(x,\frac{h}{2}\right) + O(h),$$

and

$$\begin{aligned} &u_{yyy}(x,0) = p_{xy}(x,0) - (f_1)_y(x,0) - u_{xxy}(x,0), \\ &(g_1)_{xx}(x,0) = -u_{xxy}(x,0) \quad \text{since} \quad u_y(x,0) = -g_1(x,0). \end{aligned}$$

Thus,

$$\begin{aligned} p_x\left(x,\frac{h}{2}\right) + \frac{h}{24} u_{yyy}\left(x,\frac{h}{2}\right) &= \frac{p^{i+\frac{1}{2},\frac{1}{2}} - p^{i-\frac{1}{2},\frac{1}{2}}}{h} + \frac{1}{24}\left(\frac{p^{i+\frac{1}{2},\frac{3}{2}} - p^{i+\frac{1}{2},\frac{1}{2}}}{h} - \frac{p^{i-\frac{1}{2},\frac{3}{2}} - p^{i-\frac{1}{2},\frac{1}{2}}}{h}\right) \\ &\quad + \frac{h}{24}(g_1)_{xx}(x,0) - \frac{h^2}{24} F_x\left(x,\frac{h}{2}\right) - \frac{h}{24}(f_1)_y(x,0) + O(h^3). \end{aligned}$$

Since $u_{xx} + u_{yy} - p_x + f_1 = 0$, by substituting these into (3.8) we obtain

$$\begin{aligned} &-\frac{u^{i,\frac{3}{2}} - u^{i,\frac{1}{2}}}{h^2} - \frac{u^{i+1,\frac{1}{2}} - 2u^{i,\frac{1}{2}} + u^{i-1,\frac{1}{2}}}{h^2} - \frac{1}{6}\left(\frac{u^{i+1,\frac{3}{2}} - 2u^{i,\frac{3}{2}} + u^{i-1,\frac{3}{2}}}{h^2} - 2\frac{u^{i+1,\frac{1}{2}} - 2u^{i,\frac{1}{2}} + u^{i-1,\frac{1}{2}}}{h^2}\right) - \frac{1}{h}\left(g_1^{i,0} + \frac{h^2}{8}(g_1)_{xx}(x,0)\right) \\ &\quad + \frac{p^{i+\frac{1}{2},\frac{1}{2}} - p^{i-\frac{1}{2},\frac{1}{2}}}{h} + \frac{1}{24}\left(\frac{p^{i+\frac{1}{2},\frac{3}{2}} - p^{i+\frac{1}{2},\frac{1}{2}}}{h} - \frac{p^{i-\frac{1}{2},\frac{3}{2}} - p^{i-\frac{1}{2},\frac{1}{2}}}{h}\right) + \frac{h^2}{24} F_x\left(x,\frac{h}{2}\right) - \frac{h}{24}(f_1)_y(x,0) \\ &\quad = f_1^{i,\frac{1}{2}} + \frac{h^2}{12}\Delta(f_1)\left(x,\frac{h}{2}\right) + O(h^3). \end{aligned} \tag{3.9}$$

Now, for the boundary condition $u(x,0) = 0$ we use the third order Taylor series expansion

$$u^{i,\frac{1}{2}} = -\frac{h}{2} g_1^{i,0} + \frac{h^2}{8} u_{yy}(x,0) + \frac{h^3}{48} u_{yyy}(x,0) + O(h^4), \tag{3.10}$$

where

$$u_{yy}(x,0) = u_{yy}\left(x,\frac{h}{2}\right) - \frac{h}{2} u_{yyy}(x,0) + O(h^2),$$

and

$$u_{yy}\left(x,\frac{h}{2}\right) = -\frac{u^{i+1,\frac{1}{2}} - 2u^{i,\frac{1}{2}} + u^{i-1,\frac{1}{2}}}{h^2} + \frac{p^{i+\frac{1}{2},\frac{1}{2}} - p^{i-\frac{1}{2},\frac{1}{2}}}{h} - f_1\left(x,\frac{h}{2}\right) + O(h^2).$$

Here, we have

$$u_{yyy}(x,0) = p_{xy}(x,0) - (f_1)_y(x,0) - u_{xxy}(x,0). \tag{3.11}$$

Note that

$$\begin{aligned} &u_{xxy}(x,0) + v_{xyy}(x,0) = 0, \\ &v_{xyy}(x,0) = -v_{xxx} - (f_2)_x + p_{xy}(x,0), \quad v_{xxx}(x,0) = 0. \end{aligned} \tag{3.12}$$

By substituting (3.12) to (3.11), we have

$$u_{yyy}(x,0) = -2u_{xxy}(x,0) - (f_1)_y + (f_2)_x. \tag{3.13}$$

From (3.10)

$$\begin{aligned} u^{i,\frac{1}{2}} &= -\frac{h}{2} g_1^{i,0} + \frac{h^2}{8} u_{yy}\left(x,\frac{h}{2}\right) - \frac{h^3}{24} u_{yyy}(x,0) + O(h^4) \\ &= -\frac{h}{2} g_1^{i,0} + \frac{h^2}{8}\left(-\frac{u^{i+1,\frac{1}{2}} - 2u^{i,\frac{1}{2}} - u^{i-1,\frac{1}{2}}}{h^2} + \frac{p^{i+\frac{1}{2},\frac{1}{2}} - p^{i-\frac{1}{2},\frac{1}{2}}}{h} - f_1\left(x,\frac{h}{2}\right)\right) + \frac{h^3}{24}(-2(g_1)_{xx} + (f_1)_y - (f_2)_x) + O(h^4). \end{aligned} \tag{3.14}$$

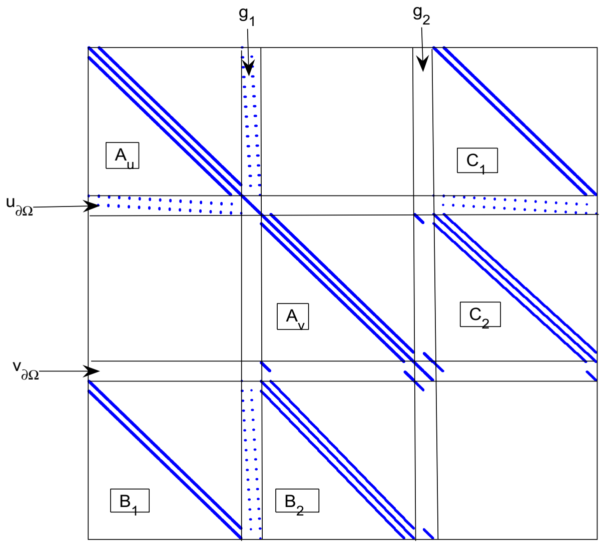


Fig. 2. The discretization matrix for the fourth order scheme with Dirichlet boundary condition on the velocity.

For the divergence free condition, note that

$$-v_{yyy}(x,\frac{h}{2}) = u_{xyy}(x,\frac{h}{2}) = \frac{\frac{u^{i+1,\frac{3}{2}}-u^{i,\frac{3}{2}}-(u^{i+1,\frac{1}{2}}-u^{i,\frac{1}{2}})}{h^2} - u_{xy}(x,0)}{h} + O(h^3).$$

Thus, at $(ih,\frac{h}{2})$, i.e., $j=0$, from (2.4) we have

$$\frac{u^{i+1,j+\frac{1}{2}}-u^{i,j+\frac{1}{2}}}{h} + \frac{1}{24h}\left(\left(u^{i+1,j+\frac{3}{2}}-u^{i,\frac{3}{2}}\right)-\left(u^{i+1,j+\frac{1}{2}}\right)-\left(u^{i,j+\frac{1}{2}}\right)\right) + \frac{1}{24}(g_1^{i+1,0}-g_1^{i,0}) + \frac{v^{i+\frac{1}{2},j+1}-v^{i+\frac{1}{2},j}}{h}$$
$$+\frac{1}{24h}\left(\left(v^{i+\frac{3}{2},j+1}-2v^{i+\frac{1}{2},j+1}+v^{i-\frac{1}{2},j+1}\right)-\left(v^{i+\frac{3}{2},j}-2v^{i+\frac{1}{2},j}+v^{i-\frac{1}{2},j}\right)\right)=0. \quad (3.15)$$

Exactly the same treatment as (3.9), (3.14) and (3.15) should be applied at $y=1$ for $-\Delta u+p_x=f_1$ and at $x=0,1$ for $-\Delta v+p_y=f_2$. So, we have a complete system for unknowns (u,g_1,v,g_2,p) for the fourth order discretization of the Stokes system (3.1). Fig. 2 shows the sketch of the discretization matrix for this system. In this figure, A_u and A_v are for the Laplacian term of u and v separately; C_1 and C_2 are for the -grad p; B_1 and B_2 are for the divergence free condition; the line block $u_{\partial\Omega}$ and $v_{\partial\Omega}$ are for the complementary boundary conditions on u and v separately; and the column block g_1 and g_2 are for the augment variables g_1 and g_2 separately.

4. Method validation

4.1. An example with exact solutions

Now we demonstrate the accuracy of our second and fourth order schemes.

The computational domain is $\Omega=[0,1]\times[0,1]$. The constructed exact solutions are

$$u=(1-\cos(2\pi x))\sin(2\pi y), \quad v=-(1-\cos(2\pi y))\sin(2\pi x),$$
$$p=\frac{1}{3}x^3. \quad (4.1)$$

So, we have the no-slip boundary condition for the velocity $\vec{u}=(u,v)$. The force $\vec{f}$ is

$$f_1=-4\pi^2(2\cos(2\pi x)-1)\sin(2\pi y)+x^2,$$
$$f_2=4\pi^2(2\cos(2\pi y)-1)\sin(2\pi x), \quad (4.2)$$

which is determined from the constructed solution and the governing Stokes equations. In Table 1, we show the grid refinement analysis by halving the mesh size in each coordinate direction. In the table, $\|E_n(\cdot)\|_\infty$ is the infinity norm of the error according to u, v and p on the corresponding staggered grid points. The order of convergence is defined as

$$\text{Order}(\cdot)=\frac{\log(\|E_{n/2}(\cdot)\|_\infty/\|E_n(\cdot)\|_\infty)}{\log 2}. \quad (4.3)$$

Table 1
The grid refinement analysis of the solutions for the constructed example

n	$\|E_n(u)\|_\infty$	Order (u)	$\|E_n(v)\|_\infty$	Order (v)	$\|E_n(p)\|_\infty$	Order (p)
4	3.3050×10^{-1}		3.3050×10^{-1}		5.2083×10^{-3}	
8	9.7985×10^{-2}	1.7540	9.7985×10^{-2}	1.7540	1.3021×10^{-3}	2.0000
16	2.5404×10^{-2}	1.9475	2.5404×10^{-2}	1.9475	3.2552×10^{-4}	2.0000
32	6.4069×10^{-3}	1.9873	6.4069×10^{-3}	1.9873	8.1380×10^{-5}	2.0000
64	1.6052×10^{-3}	1.9969	1.6052×10^{-3}	1.9969	2.0345×10^{-5}	2.0000

Second order convergence is confirmed.

For our second and fourth order schemes, the claimed order can be seen clearly. Moreover, we did the same analysis for u_x and v_y with the second order method in Table 2. Note that these quantities are defined at the mesh center points so as p in our staggered grid setup. Table 2 shows that they are of the second order and also Table 1 for the pressure p is of the full order. The standard analysis may only yield to first order convergence for these quantities. It means that our second order scheme has a super convergence property of the quantities defined at the grid-points. Table 3 shows that the pressure p is of fourth order. It is expected that the super convergence property is valid for u_x and v_y for the fourth order scheme. But for such analysis, we must use high order interpolations of u_x and v_y. We also tested both schemes for the rectangular domains with the exact solutions, similar to (4.1), and we have observed the desired order.

4.2. Driven cavity in rectangular domains

We consider a rectangular driven cavity of aspect ratio (width/height) $A = 0.75$ and $r \equiv U_A/U_B = 1$, which is taken from [7] with boundary conditions:

$$u = \begin{cases} 0, & x = 0, \quad A \\ 1, & y = 0, \quad 1 \end{cases} \tag{4.4}$$
$$v = 0.$$

Here U_A is the velocity value on the top wall, and U_B is the velocity value on the bottom wall.

In Fig. 3 we present our results for the internal velocity field for $A = 0.75$ with $h = \Delta x = \Delta y = \frac{1}{20}$. It shows the separations of up and down circulation flows due to the same directional boundary velocity. As tested in (4.4), we also use our second and fourth order methods for the rectangular driven cavity flow for $A = 0.75$ and $A = 0.25$ with a different set of boundary velocities: $U_A = U_B = 1;\ U_A = 1, U_B = -1$, which were reported in [8,10]. In Fig. 4 we present the four different cases. Our resulting flows match very well with those reported in [8,10]. Figures are for the fourth order method. And they are generated by matlab routine *streamline*(x, y, u, v, sx, sy).

In Table 4 we report for an error analysis for the second order method for the square driven cavity. Since the exact solution has singularities at the four corners (i.e., u, v do not belong to $H^1(\Omega)$ and p is a distribution.) due to the boundary

Table 2
The grid refinement analysis of the derivative of the velocities for the constructed example

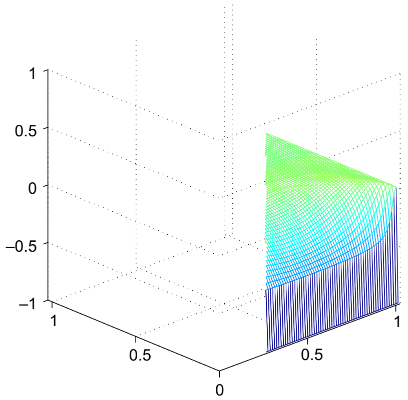
n	$\|E_n(u_x)\|_\infty$	Order (u_x)	$\|E_n(v_y)\|_\infty$	Order (v_y)
4	3.4784×10^{-1}		3.4784×10^{-1}	
8	1.4036×10^{-1}	1.3093	1.4036×10^{-1}	1.3093
16	3.9012×10^{-2}	1.8472	3.9012×10^{-2}	1.8472
32	1.0007×10^{-2}	1.9628	1.0007×10^{-2}	1.9628
64	2.5179×10^{-3}	1.9908	2.5179×10^{-3}	1.9908

Second order convergence is confirmed.

Table 3
The grid refinement analysis of the solutions for the constructed example

n	$\|E_n(u)\|_\infty$	Order (u)	$\|E_n(v)\|_\infty$	Order (v)	$\|E_n(p)\|_\infty$	Order (p)
4	6.5634×10^{-2}		6.5622×10^{-2}		9.8625×10^{-2}	
8	2.9419×10^{-3}	4.4796	2.9403×10^{-3}	4.4801	8.7720×10^{-3}	3.4910
16	1.1569×10^{-4}	4.6684	1.1556×10^{-4}	4.6692	1.6138×10^{-3}	2.4424
32	4.4844×10^{-6}	4.6892	4.4792×10^{-6}	4.6893	1.7495×10^{-4}	3.2055
64	2.1626×10^{-7}	4.3741	2.1596×10^{-7}	4.3744	1.7326×10^{-5}	3.3361

Fourth order convergence is confirmed.



5. Conclusions

In this paper, fourth order MAC finite difference schemes based on the staggered grid are developed to solve the Stokes equations with the Dirichlet boundary condition on the velocity. We introduce the higher order boundary treatment utilizing the augmented variable (boundary flux) and complete the system. For the corresponding second order method in 3.1, we have a complete stability and convergence analysis. We are still working on the stability and convergence analysis for the fourth order scheme. Accuracy of the method is verified through test problems. Separated flow can be seen from the driven cavity simulation, which was shown by the published papers experimentally and numerically.

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